Indian Statistical Institute
Back Paper Examination
Algebra IV - BMath II year.

Max Marks: 100 Time: 3 hours.

Answer all questions. Throughout, $\mathbb{Q}$ denotes the field of rational numbers, and $\mathbb{F}_p$ denotes the field with p elements.

(1) Determine the splitting field K of the polynomial $x^6 - 4$ over $\mathbb{Q}$, and the degree of the extension $K/\mathbb{Q}$. [10]

(2) (a) Define *perfect fields*.
(b) Show that every irreducible polynomial over a perfect field is separable.
(c) Give an example of an irreducible polynomial, which is inseparable, over a field which is not perfect. [2+8+5]

(3) (a) Given a prime p and a positive integer n, show that, upto isomorphism, there exists a unique field $\mathbb{F}_{p^n}$, of order p^n.
(b) Show that the extension $\mathbb{F}_{p^n}/\mathbb{F}_p$ is a Galois extension.
(c) Determine the Galois group. [10+5+5]

(4) Prove that if K_1, K_2 are Galois extensions of a field F, then the composite K_1K_2 over F is Galois with Galois group a subgroup of the direct product $Gal(K_1/F) \times Gal(K_2/F)$. [15]

(5) Prove that $\mathbb{F}_p(x, y)/\mathbb{F}_p(x^p, y^p)$ is not a simple extension, for a prime p, and x, y indeterminates. [15]

(6) Prove that a regular n-gon is constructible if and only if $\phi(n)$ is a power of 2, where ϕ denotes the Euler's phi-function. [15]

(7) Let K/F be a Galois extension with Galois group, the symmetric group S_n. Show that K is the splitting field of an irreducible polynomial of degree n in $F[x]$. [10]